J. Fluid Mech. (1967), *vol.* 28, *part* 1, *pp.* 149–151
Printed in Great Britain

Three-dimensional flow near a two-dimensional stagnation point

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(Received 24 August 1966)

This paper shows the existence of a three-dimensional solution of the boundary-layer equations of viscous incompressible flow in the immediate neighbourhood of a two-dimensional stagnation point of attachment. The numerical solution has been obtained.

The flow of a viscous incompressible fluid near a stagnation point of attachment has been shown by Howarth (1951) to be of a simple form. Let P be such a stagnation point at a regular point of the surface of the body, and suppose that the external flow is irrotational. Then near P the body may be represented by its tangent plane and a rectangular Cartesian co-ordinate system (x, y, z) may be chosen with the x, y-axes in this plane so that the external flow near P has components $\{ax, by, -(a+b)(z-\delta)\}$. We choose the x-axis so that $a > 0$; b may be any positive or negative constant such that $a+b > 0$, and δ is the three-dimensional boundary-layer displacement thickness.

Howarth expressed the velocity components (u, v, w) of the flow in the boundary layer in the form

$$u = axf'(\eta), \quad v = byg'(\eta), \quad w = -\nu^{\frac{1}{2}}\{af(\eta)+bg(\eta)\}/a^{\frac{1}{2}}, \tag{1}$$

where $\eta = a^{\frac{1}{2}}z/\nu^{\frac{1}{2}}$ is the usual boundary-layer variable. The continuity equation is satisfied by (1) and the boundary-layer equations require that f, g satisfy the following pair of ordinary differential equations, namely

$$f''' + (f+cg)f'' + 1 - f'^2 = 0, \tag{2}$$

$$cg''' + (f+cg)cg'' + c^2 - c^2g'^2 = 0, \tag{3}$$

where $c = b/a$.

The boundary conditions for (2), (3) are

$$\left.\begin{aligned} f = g = f' = g' = 0 \quad &\text{when} \quad \eta = 0, \\ f' \to 1, \quad g' \to 1 \quad &\text{as} \quad \eta \to \infty. \end{aligned}\right\} \tag{4}$$

The solution given by (2), (3), (4) is a full solution of the Navier–Stokes equations. When $c = 1$ then $f = g$ and we have the solution for the flow at an axi-symmetrical stagnation point; when $c = 0$ then $b = 0$ so that $v = 0$ and we have the flow at a two-dimensional stagnation point. There is, however, another solution in the neighbourhood of $c = 0$.

Let us expand f, g in the form

$$f = \sum_{n=0}^{\infty} c^n f_n(\eta), \quad g = \sum_{n=-1}^{\infty} c^n g_n(\eta), \tag{5}$$

where f_n, g_n are independent of c.

To satisfy (4) for all values of c then

$$f'_{n+1} \to 0, \quad g'_n \to 0 \quad \text{as} \quad \eta \to \infty \quad (n \geqslant 0). \tag{6}$$

We have introduced a simple pole at $c = 0$ in g, but cg which is the physically important quantity, has no singularity. If we use (5) in (2), (3) and take the limit as $c \to 0$ we obtain the following equations for f_0 and g_{-1}

$$f_0''' + (f_0 + g_{-1}) f_0'' + 1 - f_0'^2 = 0, \tag{7}$$

$$g_{-1}''' + (f_0 + g_{-1}) g_{-1}'' - g_{-1}'^2 = 0. \tag{8}$$

The boundary conditions are

$$\left. \begin{aligned} f_0 = g_{-1} = f_0' = g_{-1}' = 0 \quad \text{when} \quad \eta = 0, \\ f_0' \to 1, \quad g_{-1}' \to 0 \quad \text{as} \quad \eta \to \infty. \end{aligned} \right\} \tag{9}$$

η	f_0	f_0'	f_0''	$-g_{-1}$	$-g_{-1}'$	$-g_{-1}''$
0	0	0	1·178	0	0	0·630
0·1	0·006	0·113	1·078	0·003	0·063	0·629
0·2	0·022	0·216	0·981	0·013	0·126	0·628
0·3	0·049	0·309	0·886	0·028	0·188	0·625
0·4	0·084	0·393	0·796	0·050	0·251	0·618
0·5	0·127	0·468	0·712	0·078	0·312	0·608
0·6	0·177	0·536	0·633	0·113	0·372	0·593
0·7	0·234	0·595	0·561	0·153	0·430	0·572
0·8	0·296	0·648	0·495	0·199	0·486	0·546
0·9	0·363	0·695	0·436	0·250	0·539	0·514
1·0	0·435	0·735	0·382	0·306	0·589	0·476
1·1	0·510	0·771	0·334	0·368	0·634	0·433
1·2	0·589	0·802	0·291	0·433	0·675	0·384
1·3	0·670	0·830	0·253	0·502	0·711	0·330
1·4	0·755	0·853	0·220	0·575	0·741	0·272
1·5	0·841	0·874	0·191	0·651	0·765	0·210
1·6	0·929	0·892	0·165	0·728	0·783	0·147
1·7	1·019	0·907	0·143	0·807	0·795	0·082
1·8	1·111	0·920	0·124	0·887	0·800	0·018
1·9	1·203	0·932	0·107	0·967	0·798	−0·046
2·0	1·297	0·942	0·092	1·046	0·790	−0·107
2·2	1·487	0·958	0·068	1·201	0·758	−0·219
2·4	1·680	0·970	0·051	1·348	0·704	−0·310
2·6	1·875	0·978	0·037	1·482	0·635	−0·375
2·8	2·071	0·985	0·027	1·601	0·556	−0·412
3·0	2·268	0·989	0·020	1·704	0·472	−0·422
3·2	2·466	0·993	0·014	1·790	0·389	−0·408
3·4	2·665	0·995	0·010	1·860	0·311	−0·374
3·6	2·864	0·997	0·007	1·915	0·240	−0·328
3·8	3·064	0·998	0·005	1·957	0·180	−0·275
4·0	3·264	0·999	0·003	1·988	0·130	−0·221
4·5	3·763	1	0·001	2·031	0·050	−0·107
5·0	4·263	1	0	2·046	0·016	−0·040
5·5	4·763	1	0	2·050	0·004	−0·012
6·0	5·263	1	0	2·051	0·001	−0·003
7·0	6·263	1	0	2·051	0	0

TABLE 1

In particular we notice from (1) that as $c \to 0$ then $v \to ayg'_{-1}(\eta)$ so that the boundary-layer flow is *three-dimensional.*

The general asymptotic expansions for f_0, g_{-1} are of the form

$$1 - f'_0 \sim A_1 e^{-\frac{1}{2}\chi^2} \chi^{-3} + B_1 \chi^2, \tag{10}$$

$$g'_{-1} \sim A^2 e^{-\frac{1}{2}\chi^2} \chi^{-1} + B_2; \tag{11}$$

where A_1, A_2, B_1, B_2 are constants and $\chi \equiv \eta - \alpha - \beta$, where α, β are respectively the limits of $(\eta - f_0), -g_{-1}$, as $\eta \to \infty$. The three-dimensional boundary-layer displacement thickness is thus $\nu^{\frac{1}{2}}(\alpha+\beta)/a^{\frac{1}{2}}$. The required solution for f_0, g_{-1} is that for which $B_1 = B_2 = 0$ so that the outer boundary condition is satisfied. This solution has $f''_0(0) = 1{\cdot}177958$ and $g''_{-1}(0) = -0{\cdot}629565$ and is given in the accompanying table.

We notice from table 1 that when $1{\cdot}2 < \eta < 5{\cdot}4$ then $f'^2_0 + g'^2_{-1} > 1$ so that when $x < y$ the speed of the boundary-layer flow is greater than that of the external flow. This is plausible as the stagnation point is a saddle-point. The non-dimensional displacement thickness $(\alpha + \beta)$ is $2{\cdot}787$, which is much larger than the corresponding value $0{\cdot}648$ of the usual two-dimensional solution. This is because the flow near the y-axis towards the stagnation point produces a blockage and the skin-friction component in the x-direction is also reduced.

This solution is probably of more mathematical interest rather than physical importance. It is, however, a finite disturbance solution which contains streamwise vorticity and thus it may have a bearing on the three-dimensional instability of two-dimensional flows.

Our thanks go to Professor P. A. Libby, whose recent work led us to discover the new solution, and also to Mr E. J. Watson for some helpful discussions. This work was done as part of the research programme of the National Physical Laboratory, and is published by permission of the Director.

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